$$\int_1^e (x\ln x)dx = \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x}{2}dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$$

D'où :

$$\int_1^e x(\ln x)^2 dx = \frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \frac{1}{4} = \frac{e^2-1}{4}$$

Conclusion :

$$\boxed{\int_1^e x(\ln x)^2 dx = = \frac{e^2-1}{4}}$$

Exercice 6

1) $$I+J = \int_0^{\frac{\pi}{2}} (e^x \cos x)^2 dx + \int_0^{\frac{\pi}{2}} (e^x \sin x)^2 dx$$

$$= \int_0^{\frac{\pi}{2}} \left[(e^x \cos x)^2 + (e^x \sin x)^2\right] dx$$

$$= \int_0^{\frac{\pi}{2}} (e^{2x}\cos^2 x + e^{2x}\sin^2 x)dx$$

$$= \int_0^{\frac{\pi}{2}} e^{2x}(\cos^2 x + \sin^2 x)dx$$

$$= \int_0^{\frac{\pi}{2}} e^{2x}dx = \left[\frac{e^{2x}}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{e^\pi}{2} - \frac{1}{2} = \frac{e^\pi - 1}{2}$$

2) $$I-J = \int_0^{\frac{\pi}{2}} (e^x \cos x)^2 dx - \int_0^{\frac{\pi}{2}} (e^x \sin x)^2 dx$$

$$= \int_0^{\frac{\pi}{2}} e^{2x}\cos^2 x dx - \int_0^{\frac{\pi}{2}} e^{2x}\sin^2 x dx$$

$$= \int_0^{\frac{\pi}{2}} e^{2x}(\cos^2 x - \sin^2 x)dx$$

On sait que: $\boxed{\cos 2x = \cos^2 x - \sin^2 x}$

D'où : $I-J = \int_0^{\frac{\pi}{2}} e^{2x}.\cos 2x dx$

Intégrant par parties:

$$U(x) = \cos 2x \to U'(x) = -2\sin 2x$$

$$V'(x) = e^{2x} \to V(x) = \frac{e^{2x}}{2}$$

D'où

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos 2x dx = \left[\frac{1}{2}e^{2x}\cos 2x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^{2x}\sin 2x dx$$

$$= -\frac{1}{2}e^\pi - \frac{1}{2} + \int_0^{\frac{\pi}{2}} e^{2x}\sin 2x dx$$

Integrant par parties $\int_0^{\frac{\pi}{2}} e^{2x}\sin 2x dx = K$

On pose:

$$U(x) = \sin 2x \to U'(x) = 2\cos 2x$$

$$V'(x) = e^{2x} \to V(x) = \frac{e^{2x}}{2}$$

Donc:

$$K = \left[\frac{e^{2x}}{2}\sin 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos 2x.e^{2x}dx$$

Par la suite $K = -\int_0^{\frac{\pi}{2}} e^{2x}.\cos 2x dx$

Il en résulte:

$$\int_0^{\frac{\pi}{2}} e^{2x}.\cos 2x dx = -\frac{1}{2}(e^\pi + 1) - \int_0^{\frac{\pi}{2}} e^{2x}.\cos 2x dx$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} e^{2x}.\cos 2x dx = -\frac{1}{2}(1+e^\pi)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} e^{2x}.\cos 2x dx = -\frac{1}{4}(1+e^\pi)$$

Ce qui donne : $\boxed{I-J = -\frac{1}{4}(1+e^\pi)}$

3)